ADRIANNA URBANO	MANDALA ILLUSIONS	SCTC 4385
Grade Band: 7th – 10th	**Topics:** Modular Arithmetic Art	**Lesson #:** 4- to 5-Day Unit Plan

BRIEF LESSON DESCRIPTION

Description:

Your role as an illusionist makes you responsible for designing and creating intricate artwork. Your specific task is to define an underlying pattern and apply reflections and/or rotations appropriately. Since you are in training, you will be using logic and problem-solving skills to learn about how to define these underlying patterns. You will create illusions and work with two or three other illusionists on a research project and presentation following your individual illusion creations.

Leading Question:

How can we use mathematics to create intricate illusions?

PERFORMANCE EXPECTATIONS

Students Will Be Able To:

- Understand how to perform modular arithmetic
- Understand and utilize the modulo notation
- Understand the connection between modulus and the theory of congruence classes
- Apply modular arithmetic and geometry (reflections and rotations) to artwork
- Conduct meaningful research
- Connect the concept of modular arithmetic to architecture

Students Will Understand:

Modular arithmetic is based in divisibility. Through this unit, students will use the properties of divisibility to learn the basics of modulus. The modulo operation results in the remainder from division. Through this unit plan, the students will use modular arithmetic to define a pattern for their artwork, apply their prior knowledge of reflections and rotations to transform the pattern into an illusion, and later conduct research to learn why and how modular arithmetic connects to architecture.

Key Definitions & Concepts:

- Division: the inverse operation of multiplication; repeated subtraction
- Modular Arithmetic: a method for finding remainders where all the possible numbers (less than the divisor) are put in a circle, and then by counting around the circle the number of times that the number is being divided, where the remainder is the final number landed on; often called “clock math” or “circle math”
- Remainder: the amount that is “left-over” after dividing one number by another

SPECIFIC LEARNING OUTCOMES

Standards:

CC.2.2.HS.C.3: Write functions or sequences that model relationships between two quantities.

- This standard connects to the unit plan because the students will be using modular arithmetic to connect one statement to another that have different representations but are the equivalent.

CC.2.2.HS.C.4: Interpret the effects transformations have on functions and find the inverses of functions.

- This standard connects to the unit plan because the students will be using modular arithmetic to define a pattern based on congruence classes. These patterns will be found through inverse functions: addition, subtraction, and multiplication.

BACKGROUND INFORMATION

Prior Knowledge:

- Difference between integers and natural numbers
- Whole number division (long division process with a whole number remainder)
- Geometric reflections about an axis and geometric rotations

Mathematical Practices:	Disciplinary Core Ideas:	Crosscutting Concepts:
• Construct viable arguments and critique the reasoning of others • Reason abstractly and quantitatively • Look for make use of structure	• Mathematical relationships among numbers can be represented, compared and communicated • Mathematical relations and functions can be modeled through multiple representations and analyzed to raise and answer questions	• Observe and make use of patterns • Identify and utilize structure and function

Possible Preconceptions/Misconceptions:

Many students will complete the division process properly, but disregard having a whole number remainder. Since the point of this unit is to learn via inquiry and discovery, then expect students to struggle to understand why they need non-negative, whole number remainders. This is a key concept that they should fully understand after the encryption introduction and lesson.

LESSON PLAN → 5E (+) MODEL
Engage: Engage the students with an anchoring event: a video clip from The Exploratorium. This clip shows an actor straddling a mirror and providing illusion-based entertainment to a crowd of people. The purpose of clip is to engage students into a discussion of how to create an illusion through reflections. The following is the link to the clip: Anti-Gravity Mirror (https://www.exploratorium.edu/video/anti-gravity-mirror-clip). The students will then be engaged in a whole class discussion about how that video clip connects with the unit plan's goals and objectives. Teachers will use student responses to lead into the RAFT of the unit: the students' role as an illusionist.
Explore: Students will work in their groups (4 to 5 members per group) to complete a worksheet that explores modular arithmetic and the importance of the remainder in divisibility processes. This guided worksheet includes the following: connecting division to time via "Clock Math" and connecting the "Clock Math" to "Circle Math." The purpose of using "Clock Math" is to provide students with examples of modulus in base 12, but in a manner that is familiar to them (i.e. gauging time). The purpose of using "Circle Math" is to extend their understanding of modulus from base 12 to modulus of any natural number base. However, the students will not be directly told that they are using modular arithmetic. The worksheet also includes a critical thinking question in which students are shown examples of the modulus operator but not told what it is. They must analyze the expressions and the respective answers to produce an explanation of modulus. They will be engaged in group- and whole-class discussions regarding the patterns within the examples, and regarding the connection between the clocks, circles, and the "mystery" (modulo) operator. The students will then be formally introduced to modular arithmetic and given a definition. They will extend the clock and circle examples into formal modulus writing, and the connection of remainders to the theory of congruence classes. These worksheets are the *benchmark lesson*: what modular arithmetic is, the theory of congruence classes, and the processes of adding, subtracting and multiplying congruence classes. Finally, the class will lead into the *investigation lesson*: exploring the connection between modulus and illusionary artwork. The students will not have a guided worksheet for this part of the lesson. Rather, they will work individually and with their groups to explore defining a pattern and completing the reflections or rotations.
Explain: Throughout the exploration, the students will engage in discussions that inquire their understanding and knowledge of the information at-hand. Thus, teachers will be informally asking students to explain all topics and relevant connections throughout the entirety of this unit. The guided worksheet contains sections asking students how they arrived at the answer and asking if they notice any patterns between the problems. The final project presentation is

designed, in part, so that the students communicate the research information to the class along with their created illusions.

Elaborate:

Concluding the exploration, the students will extend their understanding of modulus and congruence classes by conducting research to answer the following: (1) how is modulus used in architecture, (2) how did geometry play a role within the illusions, and (3) where else is modulus seen, and detail an example. The purpose of having the students answer these questions through research is to give them opportunity to make their presentations unique to their interests within the scope of this unit plan. This portion is the conclusion of the *investigation lesson* and the last section of the summative assessment.

Evaluate:

This unit plan is designed with having both informal and formal evaluations throughout its entirety. The informal evaluations will occur on all four or five days and in multiple outlets. Predominantly, the students will be asked leading and open-ended questions throughout the exploration of the unit. This will allow teachers to gauge surface-level student understanding. The guided worksheets provide the students with an outline of steps to understanding modulus and congruence classes. By surveying the students during completion of these worksheets, teachers will be able to hear any misconceptions or misunderstandings, and address such immediately. Also, teachers will be able to gauge the percentage of students grasping the content by listening to their conversations and explanations within the groups.

The formal evaluation within this unit plan occurs on the final day, and it includes the formal presentation and the individual illusions. Throughout the unit plan, the students will have had ample time to grapple with modulus and congruence classes, ask questions, answer discussion topics and extend their understanding through research. The purpose of the presentation is to have the students communicate what they learned about the connection between modulus to architecture, the connection from geometry to their illusion, and where else modulus is used to their peers. Each presentation is expected to be between 15 and 20 minutes long per group. Since there are 4 to 5 members per group, each student is expected to between a four (4) to five (5) minute part within their group's total presentation.

Enrichment:

This lesson plan can be differentiated by teaching the students cryptography. Modulus is widely used in computer science, especially for encryption and decryption keys. This lesson could be extended into a computer science class by having students learn cipher methods, and even write code that completes a specific cipher's encryption or decryption process.

See all attachments on the following pages

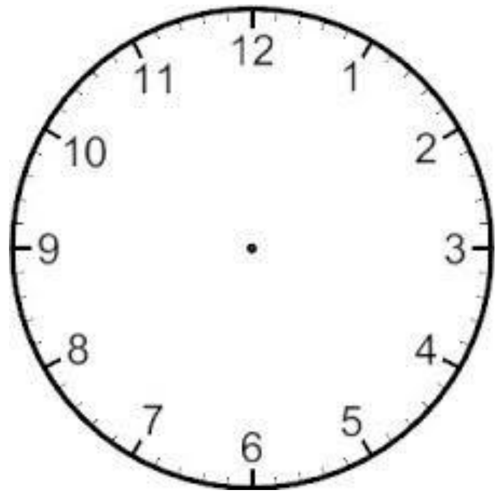
Rubric for Content Knowledge and Understanding

Criteria	Developing (does not meet performance standards)	Proficient (meets performance standards)	Advanced (exceeds performance standards)
Part I: Illusion			
Completeness and Accuracy of Required Information : (1) The base of modulus chosen to determine the size of the illusion; (2) Proper arithmetic solved on the congruence classes; (3) The reflections/rotations applied.	One or more pieces of required information are omitted	All required information is included	All required information is included
	Significant information is incorrect, key terms are used inappropriately, and/or important details are missing	Most significant information is correct, although there are some minor errors or missing details	All information is correct and discussed in detail
Part II: Research			
Completeness and Accuracy of Required Information : (1) How is modulus used in architecture; (2) How did geometry play a role within the illusions; (3) Where else is the modulus seen, and provide a detailed example.	One or more pieces of required information are omitted	All required information is included	All required information is included
	Significant information is incorrect, key terms are used inappropriately, and/or important details are missing	Most significant information is correct, although there are some minor errors or missing details	All information is correct and discussed in detail
Part III: Presentation			
Completeness and Accuracy of Required Information : (1) Detailed explanation of the development of the Mandalas; (2) Detailed explanation of the research content	Less that half of the group members do not present a portion of the group's information	Half of the group members do not present a portion of the group's information	All group members present with equal speaking time
	Presentation is shorter than 5 minutes or longer than 25 minutes	Presentation is shorter than 15 minutes or longer than 20 minutes	Presentation is within the 15 to 20 minute time window
	Presenters were not able to accurately answer the posed questions after their given presentation	Presenters provided answers to posed questioning with few incorrect assumptions or few misunderstood concepts after their given presentation	Presenters provided knowledgable and coherent answers to posed questioning after their given presentation

Name: .______________________________________.   Date______________________.

Clock Math

1. Say it is 9:00 AM, what time will it be in:
    a. 20 hours? How did you figure this out?



    b. 55 hours? How did you figure this out?

    c. 81 hours? How did you figure this out?

    d. Do you notice any patterns?

Name: ____ Date: ______________________

Circle Math

2. Use a circle to figure out (do not use traditional division):

58/9 =

35/4 =

18/10 =

100/19 =

Are there any similarities between this problem and problem #1?

Name: ____ Date: ______________________

Modular Arithmetic

3. Now consider the "mystery" operation denoted by "mod":

58mod9 = 4
18mod10 = 8
35mod4 = 3
20mod12 = 8

Do you notice any similarities between this operation and what you did in problems #1 and #2?

Hypothesize what the mod operation does:

Name: ____ Date: ______________________

Congruence Classes

4. Fill in the blanks:

   a. To do modular arithmetic, we first need to have a modulus. When we divide an integer by this modulus, the result is the _____________.

   b. The result of modulus is always non-_________ and is __________ than the modulus.

   c. Let the modulus be 12. The only possible remainders are the integers ________.

5. Concept Challenge!

   Let the modulus be any integer *n*.
   Then we have the representation $a \equiv b \bmod n$ where *a* and *b* are integers, and where *a* is the resulting remainder upon dividing *b/n*.
   What are the only possible reminders? i.e., what is the set of values that *a* can take on?

6. List the first five values that are in the following classes:

   a. $[0]_{12}$:

   b. $[1]_3$:

   c. $[4]_4$:

   d. $[3]_2$:

   e. $[0]_5$:

   f. $[2]_4$:

   g. $[5]_6$:

Name: ____ Date: ______________________

Congruence Arithmetic

7. Fill in the addition table for modulus 4:

ADDITION	[0]	[1]	[2]	[3]
[0]				
[1]				
[2]				
[3]				

8. Fill in the subtraction table for modulus 3:

SUBTRACTION	[0]	[1]	[2]
[0]			
[1]			
[2]			

9. Fill in the multiplication table for modulus 5:

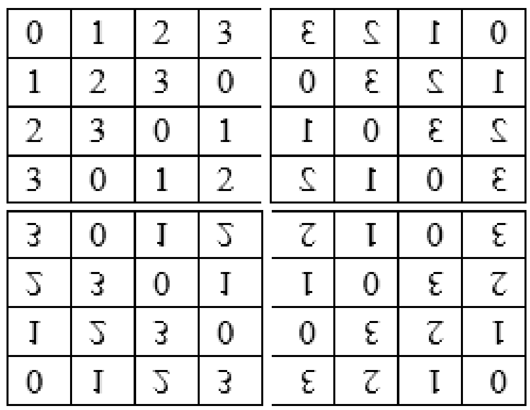
MULTIPLICATION	[0]	[1]	[2]	[3]	[4]
[0]					
[1]					
[2]					
[3]					
[4]					

Name: ____ Date: _______________________

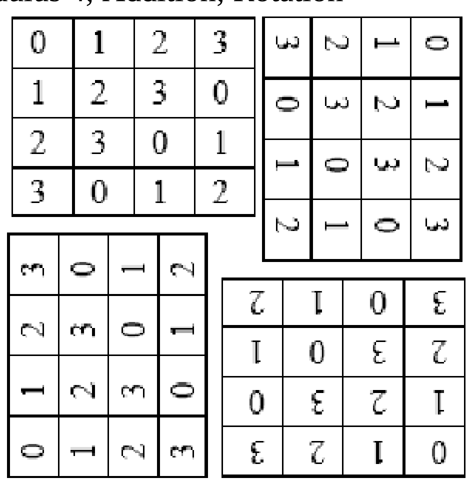
Mandala Illusions

<u>Directions:</u> Everything covered thus far led us to this moment – your practice as an illusionist. Using the theories that we covered, create your own Mandala. The following are the minimum requirements:

1. Minimum of a modulus 7 defined pattern.
2. Minimum of two mandala illusions – each using a different congruence arithmetic, and each using the modulus chosen in step 1.
3. Apply a reflection to one of your pieces, and apply a rotation to the other piece.
    a. Example: Modulus 4; Addition; Reflection



    b. Example: Modulus 4; Addition; Rotation

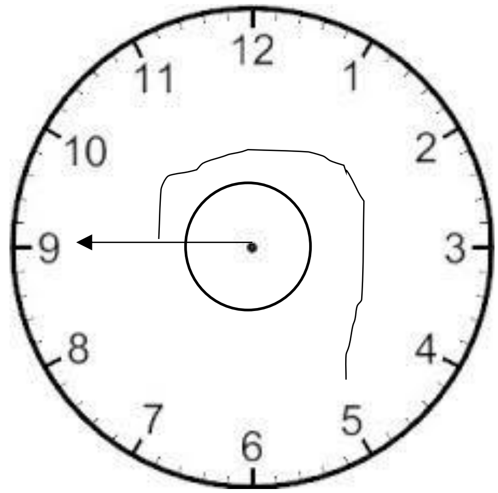


4. Incorporate individual characteristics – create your own designs within the boxes. Make this part colorful and intricate!
5. Have fun!

Rulers and pencils are necessary for completing the proper measurements and outlines of your Mandalas (you can see the necessity of perfect squares for the boxes in example b)

Name: _______________ANSWER_KEY_____________ Date: ________________________

Clock Math



1. Say it is 9:00 AM, what time will it be in:
   b. 20 hours? How did you figure this out?

      5am
      Answers may vary

   c. 55 hours? How did you figure this out?

      4pm
      Answers may vary

   d. 81 hours? How did you figure this out?

      6pm
      Answers may vary

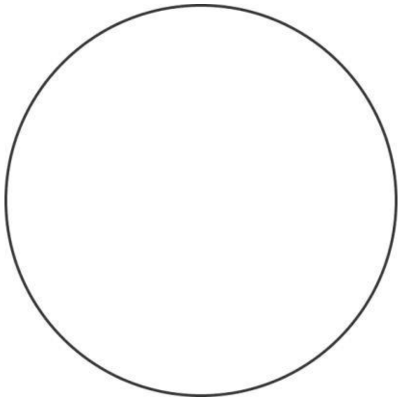
   e. Do you notice any patterns?

      Target answer: Each time we divided by 12 to see how many times the hour hand made a full rotation. Then, the remainder when we divided the time passed by 12 gave us the hour.

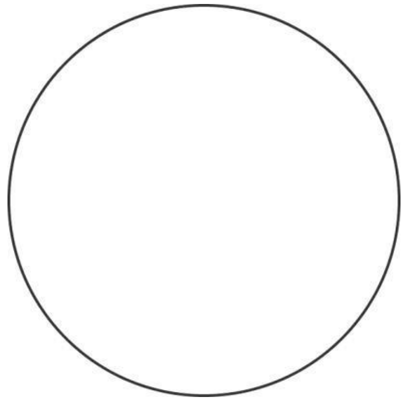
Circle Math

2. Use a circle to figure out (do not use traditional division):
   Like the clock, situate [0, base] around the clock, and make necessary rotations

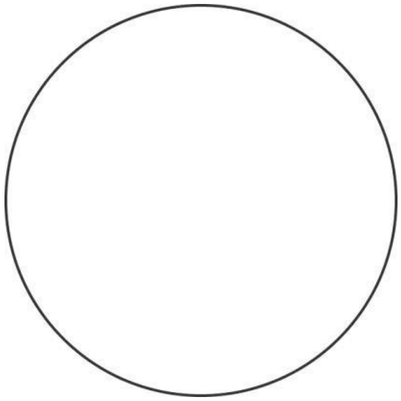
58/9 = 6 R 4



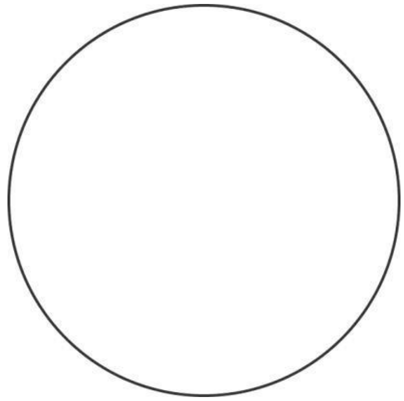
35/4 = 8 R 3



18/10 = 1 R 8



100/19 = 5 R 5



Are there any similarities between this problem and problem #1?

Here, the quotient is the number of rotations around the circle; in the clocks, the quotient was the number of rotations of the hour hand around the clock. In both, we found the remainder by seeing how many units were left over once we divided. The clocks were like this problem but we divided by 12 and had to start at a certain point.

Modular Arithmetic

3. Now consider the “mystery” operation denoted by “mod”:

   58mod9 = 4
   18mod10 = 8
   35mod4 = 3
   20mod12 = 8

   Do you notice any similarities between this operation and what you did in problems #1 and #2?

   > 58mod9 = 4 and 58/9 = 6 R 4, so they both have a 4 in the answer
   > 18mod10 = 8 and 18/10 = 1 R 8, so they both have an 8 in the answer
   > 35mod4 = 3 and 35/4 = 8 R 3, so they both have a 3 in the answer
   > 20mod12 = 8, which is how many hours we added in the first clock example after one rotation.

   > The answer between the clocks, circle and modulus is the remainder after division.

   Hypothesize what the mod operation does:

   > It gives us the remainder when the first number is divided by the second.

Name: ____________ANSWER-KEY______________ Date: ________________________

Congruence Classes

4. Fill in the blanks:

a. To do modular arithmetic, we first need to have a modulus. When we divide an integer by this modulus, the result is the ___REMAINDER___.

b. The result of modulus is always non-_NEGATIVE_ and is _LESS_ than the modulus.

c. Let the modulus be 12. The only possible remainders are the integers _0 – 11_.

5. Concept Challenge!

Let the modulus be any integer *n*.
Then we have the representation $a \equiv b \bmod n$ where *a* and *b* are integers, and where *a* is the resulting remainder upon dividing *b/n*.
What are the only possible reminders? i.e., what is the set of values that *a* can take on?

Answer: $\{0, 1, \ldots, n\text{-}1\}$

6. List the first five elements that are in the following classes:

a. $[0]_{12}$: 0, 12, 24, 36, 48, …

b. $[1]_{3}$: 4, 7, 10, 13, 16, …

c. $[4]_{4}$: 0, 4, 8, 12, 16, …

d. $[3]_{2}$: 1, 3, 5, 7, 9, …

e. $[0]_{5}$: 0, 5, 10, 15, 20, …

f. $[2]_{4}$: 2, 6, 10, 14, 18, …

g. $[5]_{6}$: 5, 11, 17, 23, 29, …

Name: ANSWER-KEY Date:

Congruence Arithmetic

Fill in the addition table for modulus 4:

ADDITION	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

Fill in the subtraction table for modulus 3:

SUBTRACTION	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[-1] ≡ [2]	[0]	[1]
[2]	[-2] ≡ [1]	[-1] ≡ [2]	[0]

Fill in the multiplication table for modulus 5:

MULTIPLICATION	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]
[2]	[0]	[2]	[4]	[1]	[3]
[3]	[0]	[3]	[1]	[4]	[2]
[4]	[0]	[4]	[3]	[2]	[1]